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ESTABLISHING THE EXISTENCE OF PURE-STRATEGY EQUILIBRIA IN SYMMETRIC TWO-PLAYER ZERO-SUM GAMES

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ABSTRACT

This paper makes two significant contributions to the study of pure equilibria in symmetric zero-sum games. First, it introduces a set of new sufficient conditions most notably the concepts of interchangeability and weak quasiconcavity that ensure the existence of such equilibria. Second, it explores how these new conditions relate to those already established in the literature. In particular, we show that weakly quasiconcave games form a broader category that includes both quasiconcave games and ordinal potential games. We also establish that exact potential games inherently satisfy the interchangeability condition. At the same time, our results indicate that there is no direct logical connection between interchangeability and (weak) quasiconcavity, highlighting the independence of these two properties.

Keywords: *Saddle Points; Symmetric Two-Player Zero-Sum Games; Pure Strategy Equilibrium; Potential Games; Quasiconcave and Weakly Quasiconcave Games; Interchangeability Conditions; Game Theoretic Optimization*

INTRODUCTION

Research on sufficient conditions that guarantee the existence of pure equilibria in zero- sum games has progressed steadily since the foundational work of Shapley (1964). Over the years, several important conditions have been identified. Notable among them is super modular games, as developed by Topkis (1998) and Milgrom and Roberts (1990); potential games, introduced by Monderer and Shapley (1996) and later extended by Voorneveld (2000); and quasiconcave games, examined in the works of Radzik (1991), Duersch et al. (2012a), and Iimura and Watanabe (2016) [1-4].

Although several of the known sufficient conditions apply to symmetric zero-sum games, the pioneering study by Duersch et al. (2012a) is the only one that explicitly leverages the symmetry structure. Their work not only utilizes symmetry but also establishes a series of equivalences among the existing conditions. Building on their contribution, this paper advances the literature in two major directions. First, we introduce new sufficient conditions, including a relaxed version of their quasiconcavity concept. Second, we develop a general framework grounded in the preference relations naturally induced by symmetric zero-sum games. This framework allows us to systematically investigate the connections between previously known conditions and the newly proposed ones. In particular, we provide a complete characterization of all pairwise relationships between the old and new conditions, as summarized in Tables 1, 2, and 3. The idea of quasiconcavity often interpreted as single-peakedness was originally introduced in the setting of zero-sum games by Radzik (1991) and later generalized by Duersch et al. (2012a). Their work demonstrated that symmetric zero-sum games satisfying quasiconcavity admit a pure equilibrium, extending Radzik's earlier result for two-player zero- sum games with quasiconcave columns and quasiconvex rows. Moreover, Duersch et al. (2012a) established that exact potential games (Monderer and Shapley 1996) and super modular games (Topkis 1998) are equivalent when viewed through the lens of symmetric zero-sum game theory [5].

This study offers two central contributions. The first, presented in **Theorem 1**, introduces a new sufficient condition termed interchangeability. This notion is inspired by classical axioms on preference relations developed by Suppes and Zinnes (1963) and Fishburn (1986). We show that this condition ensures the existence of pure equilibria in symmetric zero-sum games. Although exact potential games are found to satisfy the interchangeability requirement, we demonstrate that this condition has no direct logical connection to quasiconcave games or to the class of "transitive" games defined in Sect. 3.2. Additionally, we establish in **Proposition 2** that ordinal potential games (Monderer and Shapley 1996) generate transitive preference structures. By contrast, quasiconcave games do not necessarily produce transitive preferences, and transitivity itself does not imply quasiconcavity [6].

Our second major contribution is encapsulated in **Theorem 2**, which proves that games satisfying weak quasiconcavity admit pure equilibria. We further show that this class of games strictly extends the families of quasiconcave games and ordinal potential games, as stated in **Proposition 1**. Beyond theoretical interest, these results also apply to symmetric non-zero-sum games such as models of oligopoly and public-goods provision through the lens of relative payoff games. Schaffer (1988, 1989) introduced the notion of evolutionary stable strategies in finite populations and demonstrated that, in symmetric games, such strategies correspond to optimal strategies in the associated zero-sum relative payoff game. Subsequent studies, including Ania (2008) and Hehenkamp et al. (2010), provide further applications of this framework. The paper is organized as follows. Section 2 outlines the model and basic definitions. Section 3 reviews existing sufficient conditions in Sect. 3.1 and introduces the new conditions in Sect. 3.2. Section 4 then examines the relationships among all conditions discussed in the paper.

The setup Let $G = (X, v)$ denote a symmetric two-player zero-sum game, where both players have the same finite set of pure actions X, and $v : X \times X \to \mathbb{R}$ is the payoff function of Player 1. When Player 1 chooses $x \in X$ and Player 2 choose $y \in X$, Player 1 and Player 2 receive $v(x, y) = -v(y, x)$ and $-v(x, y) = v(y, x)$, respectively. Therefore, the payoff matrix of a player is skew-symmetric, and the players' payoffs on the main diagonal are zero [7].

Definition 1 Pure equilibrium In a symmetric two-player zero-sum game (X, v), a pair of pure actions $(x^*, y^*) \in X \times X$ is a **pure equilibrium** if $v(x^*, y^*) = \max_{x \in X} v(x, y^*) = \min_{x \in Y} v(x^*, y)$.

A pure equilibrium (x^*, y^*) is called symmetric if $x^* = y^*$. Note that in a symmetric two-player zero-sum game, a pure equilibrium exists if and only if a symmetric pure equilibrium exists. In particular, if (x^*, y^*) is an equilibrium, then so are (x^*, y^*) , (x^*, x^*) , and (y^*, y^*) . Here, x^* and y^* are called optimal strategies, $v(x^*, y^*)$ is referred to as the value, and the pair $(v(x^*, y^*), -v(x^*, y^*))$ is known as a saddle point. In symmetric two-player zero- sum games, the value is zero, since players have identical strategic opportunities and their payoffs add up to zero [8].

Let $D = (X, \succeq)$ denote a decision problem, where X is a finite set of alternatives, and the relation $\succeq \subseteq X \times X$ represents the preferences of the decision maker on X. These preferences are assumed to be complete, that is, for any pair (x, y) in $X \times X$, either $x \succeq y$ or $y \succeq x$ holds.

Definition 2 Maximal element: In a decision problem $(X, \succeq)$, an alternative $x^* \in X$ is a **maximal element** if $x^* \succeq y$ for all $y \in X$.

Given that the preference relation in a decision problem is assumed to be complete but not necessarily transitive, it is generally not possible to represent preferences by a one-variable order-preserving utility function. For such decision problems, a more viable approach for functional representation of the preferences is to use a two-variable function.2 A function $u : X \times X \to \mathbb{R}$ represents the relation $\succeq$ if for all $x, y \in X$, $u(x, y) > 0$ if and only if $x \succ y$. The function u (x, y) can be interpreted as quantifying the intensity of preference for x over y. If this intensity is greater than zero, then x is preferred to y. By using the two-variable functional representation, any symmetric two-player zero-sum game G = (X, v) uniquely induces an equivalent decision problem $D_G = (X, \succeq)$, where the payoff function v represents the preference relation $\succeq$.

Conversely, for every decision problem $D = (X, \succeq)$, there are (uncountably) many functions v that represents $\succeq$. We denote by G_D the class of symmetric two-player zero-sum games, such that for each game $G \in G_D$, D is its equivalent decision problem.

We are now ready to present a useful lemma.

Lemma 1 (i) If x^* is a maximum element in decision problem D, then (x^*, x^*) is a pure equilibrium of every symmetric two-player zero-sum game $G \in G_D$. (ii) If (x^*, x^*) is a pure equilibrium of a symmetric two-player zero-sum game G, then $x*$ is a maximal element in its equivalent decision problem D_G.

Proof (i) Let x^* be a maximum element of the decision problem $D = (X, \succeq)$. Given that $\succeq$ is complete, we have $x^* \succeq y$ for all $y \in X$. For any function v that represents the preference relation $\succeq$, it follows that $v(x^*, y) \geq 0$ for all $y \in X$. Thus, for any game $G \in G_D$ where D is its equivalent decision problem, the pure action x^* guaran-tees a payoff of 0. Therefore, x^* is an optimal strategy for all games in G_D. By symmetry, the pair (x^*, x^*) is a pure equilibrium of any game in G_D. (ii) Let (x^*, x^*) be a pure equilibrium of the game G = (X, v). Then, the value of the game is 0, meaning that the pure action x^* guarantees the payoff of 0: $v(x^*, y) \geq 0$ for all $y \in X$. For its equivalent decision problem D_G, this implies that $x^* \succeq y$ for all $y \in X$. Thus, x^* is a maximal element of D_G [8-10].

Sufficient conditions: In this section, we provide the definitions of some well-known properties that guarantee the existence of a pure equilibrium in symmetric two-player zero-sum games. Subsequently, we use Lemma 1 to introduce novel sufficiency conditions for the existence of pure equilibria.

WELL-KNOWN SUFFICIENT CONDITIONS

We begin by introducing the concept of potential games, initially developed by Maunderer and Shapley (1996). We proceed to outline two distinct variations of this seminal concept.

Definition 3 Potentials: Let (X, v) be a symmetric two-player zero-sum game.
Then

1. a function $P : X \times X \rightarrow \mathbb{R}$ is an exact potential for the game (X, v), if for every $y \in X$ and for every $x, x' \in X$
   $v(x, y) - v(x', y) = P(x, y) - P(x', y) = P(y, x) - P(y, x')$.
2. a function $P : X \times X \rightarrow \mathbb{R}$ is an ordinal potential for the game (X, v), if for every $y \in X$ and for every $x, x' \in X$,
   $v(x, y) - v(x', y) > 0 \Leftrightarrow P(x, y) - P(x', y) > 0 \Leftrightarrow P(y, x) - P(y, x') > 0$.

A game (X, v) is referred to as an exact potential game or an ordinal potential game when it admits an exact or ordinal potential function, respectively. Monderer and Shapley (1996) established that both types of potential games always admit at least one pure-strategy equilibrium. Although several broader categories of potential games have been introduced in the literature such as generalized ordinal potential games (Monderer & Shapley, 1996), best-response potential games (Voorneveld, 2000), and pseudo-potential games (Dubey et al., 2006; see also Schipper, 2004) these variants are not the focus of the present discussion. These extended classes typically impose structural conditions on particular types of actions (for example, best responses or strict best responses) rather than on the full action set. As a result, the associated decision problems for games satisfying these weaker conditions may fail to meet standard preference requirements, such as transitivity or acyclicity. An illustration of this issue is provided in Example 9 of Duersch et al. (2012a, p. 557) [10-14].

Duersch et al. (2012b) introduced the concept of a generalized rock–paper–scissors (gRPS) game and demonstrated that this structure provides both a necessary and sufficient condition for the “imitate-if-better” behavioural rule to be vulnerable to a money-pump scenario. Furthermore, they showed that any symmetric two-player zero-sum game that does not fall into the gRPS category must admit at least one pure-strategy equilibrium.

Definition 4 A symmetric two-player zero-sum game (X, v)is said to be a **generalized rock–paper–scissors (gRPS) game**

If there exists a pair ($\bar{X}$, $\bar{v}$) with $\bar{X} \subseteq X$ and $\bar{v}(x, y) = v(x, y)$for every $x, y \in \bar{X}$, such that the following condition holds: for every action $y \in \bar{X}$, there exists some action $x \in \bar{X}$ for which $v(x, y) > 0$. In other words, within the restricted action set $\bar{X}$, each action can be strictly beaten by at least one other action.

Next, we present the notion of quasiconcave games, a concept originally introduced by Radzik (1991). This class of games may be viewed as a discrete analogue of the quasiconcavity property typically defined for continuous payoff functions.

Definition 5 A symmetric two-player zero-sum game (X, v)is said to be quasiconcave if there exists a total order <on the action set X such that, for any actions $x', x, x'', y \in X$ with

$$x' < x < x'',$$

the payoff function satisfies

$$v(x, y) \geq \min \{ v(x', y),\ v(x'', y)\}.$$

This condition ensures that, with respect to the chosen ordering, the payoff to an intermediate action x is never lower than the minimum payoff of the two extreme actions x' and x'', thereby capturing a discrete form of quasiconcavity [14-17].

New Sufficient Conditions: Building on the equivalence between symmetric two-player zero-sum games and decision problems established in Lemma 1, we translate central concepts from decision theory into the game-theoretic setting. In particular, we examine fundamental properties of preference relations most notably transitivity and acyclicity and use these properties to formulate new sufficient conditions that guarantee the existence of pure-strategy equilibria in symmetric two-player zero-sum games.

Definition 6 Transitive Games: Consider a decision problem D = (X, ⪰). The preference relation ⪰on the action set X is said to be transitive if, for all $x, y, z \in X$,

$$x \succeq y \text{ and } y \succeq z \Rightarrow x \succeq z.$$

A game $G = (X, v) \in \mathcal{G}_D$ is called transitive if, in its corresponding decision-problem representation, the induced preference relation ⪰satisfies this transitivity property.

Definition 7 A cyclic Games: Let D = (X, $\succeq$) be a decision problem. The preference relation

$\succeq$on the set X is said to be **acyclic** if it contains no preference cycle; that is, for every finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$, it is not the case that.

Lemma 2 Let (X, v) be a symmetric two-player zero-sum game. If the game (X, v) is **acyclic**, then it necessarily admits at least one **pure-strategy equilibrium**.

Proof. If the game (X, v) is acyclic, then the corresponding preference relation $\succeq$ in its equivalent decision-problem formulation is also acyclic. Acyclicity ensures that the set X contains at least one maximal element, say $x^* \in X$, with respect to $\succeq$. By Lemma 1, this maximal element directly induces a pure-strategy equilibrium in the original game. Therefore, the action profile (x^*, x^*) constitutes a pure equilibrium of the symmetric two-player zero-sum game [17-20].

Since transitivity implies acyclicity, we obtain the following corollary

Corollary 1 Let (X, v) be a symmetric two-player zero-sum game. If the game (X, v) is **transitive**, then it necessarily admits at least one **pure-strategy equilibrium**.

We now introduce a definition inspired by Axiom C.3 from Fishburn (1986), which concerns structural properties of preference relations. In Fishburn's original framework, this axiom—together with several complementary conditions was employed to ensure that a particular relation on the set X forms a weak order. Since our purpose here is to derive a sufficient condition for the existence of pure equilibria in symmetric two-player zero-sum games, we adapt this axiom to a form that is appropriate for our setting. The corresponding version, tailored to our game-theoretic framework, is presented below [20-23].

Definition 8 Interchangeability Condition: Consider a symmetric two-player zero-sum game (X, v). The game is said to satisfy the interchangeability condition, or to be interchangeable, if the following holds: for all $x, y, x', y' \in X$ with $x \neq y$ and $x' \neq y'$,

$$V(x, y) > v(x', y') \Rightarrow v(x, x') \geq v(y, y').$$

In other words, whenever one action pair (x, y) yields a strictly higher payoff than another pair (x', y'), the payoff comparison between the corresponding "interchanged" pairs (x, x') and (y, y') must preserve this ordering. This condition captures a structured consistency in how payoffs respond to the swapping of actions.

Lemma 3 Let (X, v) be a symmetric two-player zero-sum game. If the game (X, v) satisfies the interchangeability condition, then the game (X, v) is acyclic.

Proof. Let (X, $\succeq$) denote the decision problem associated with the game (X, v). If the set X contains only one or two elements, the statement follows immediately. Hence, assume that $|X| \geq 3$. Our goal is to show that the induced preference relation $\succeq$ is acyclic. Lemma 2 will then ensure the existence of a maximal element, which corresponds to a pure-strategy equilibrium of (X, v).

Suppose, for the sake of contradiction, that ≥contains a strict preference cycle. That is, there exists a sequence of elements

$$x_1 \succ x_1 \succ \cdots \succ xx_{n-1} \succ x_n \text{ and } x_n \succ x_1.$$

By the definition of the interchangeability condition, for any index $i \in \{1,2, \dots, n-3\}$, the relations $x_i \succ x_{i+1} \succ x_{i+2} \succ x_{i+3}$ imply $x_i \succ x_{i+3}$. This follows from the fact that

$$v(x_i, x_{i+1}) \geq v(x_{i+3}, x_{i+2})$$

implies

$$v\,(x_i, x_{i+3}) \geq v\,(x_{i+1}, x_{i+2}) > 0,$$

hence $x_i \succ x_{i+3}$. Applying this step repeatedly yields $x_1 \succ x_4$. Furthermore, since

$$x_4 \succ x_5 \succ x_6,$$

the same reasoning gives $x_1 \succ x_6$. Continuing in this manner, we obtain

$$x_1 \succ x_k \text{ for every even } k \leq n.$$

If nis even, then this includes $x_1 \succ x_n$. But since the cycle also contains $x_n \succ x_1$, we obtain

$$x_1 \succ x_n \succ x_1$$

which is impossible.

If n is odd, the argument gives $x_1 \succ x_{n-1},\ x_{n-1} \succ x_n \succ x_1$

since $n - 1$ is even. Combined with we obtain the shorter cycle

$x_1 \succ x_{n-1} \succ x_n \succ x_1$.

Repeating the interchangeability reasoning on this 3-cycle eventually yields $x_1 \succ x_1$, which is a contradiction.

Since any assumed cycle leads to a logical contradiction, the preference relation ≥ must be acyclic. By Lemma 2, the game therefore has a maximal element, which corresponds to a pure-strategy equilibrium of (X, v) [25-30].

Theorem 1 Let (X, v) be a symmetric two-player zero-sum game. If (X, v) satisfies the **interchangeability condition**, then the game admits at least one **pure-strategy equilibrium**. **Proof.** By Lemma 3, if the game G = (X, v) satisfies the interchangeability condition, then the induced preference relation $\succeq$ in the corresponding decision problem $D_G = (X, \succeq)$ is acyclic. Applying Lemma 2, it follows that the game G = (X, v) possesses at least one pure-strategy equilibrium. While the notion of quasiconcavity introduced by Duersch et al. (2012a) leverages the symmetry of the game, it does not fully utilize the zero-sum property of the payoff structure. To address this limitation and more effectively exploit the zero-sum nature, we now introduce the concept of weak quasiconcavity.

Definition 9 Weakly Quasiconcave Games: A symmetric two-player zero-sum game (X, v) is said to be weakly quasiconcave if there exists a total order <on X such that, for every $x, x', x'', y \in X$ with

$$x' < x < x'',$$

the following condition holds:

$$\operatorname{sgn}(v(x, y)) \geq \min\{\operatorname{sgn}(v(x', y)),\ \operatorname{sgn}(v(x'', y))\},$$

where sgn (·) denotes the sign function.

As we will show, weak quasiconcavity not only generalizes the notion of quasiconcavity (see Proposition 3) but also provides a sufficient condition for the existence of pure-strategy equilibria (Theorem 2). To establish these results, we first present a lemma that connects weakly quasiconcave games with the acyclicity of their associated decision problems. This link forms a crucial step in demonstrating that weak quasiconcavity guarantees the existence of a pure equilibrium [27].

Lemma 4 Let (X, v) be a symmetric two-player zero-sum game. If the game (X, v) is **weakly quasiconcave**, then its associated preference relation in the equivalent decision problem is **acyclic**. Consequently, the game (X, v) itself is acyclic.

Proof. First, observe that the games (X, v) and (X,sgn(v)) share the same equivalent decision problem $(X, \succeq)$, since for any $x, y \in X$,

$$v(x, y) > 0 \text{ if and only if } \operatorname{sgn}(v(x, y)) > 0.$$

Now, suppose, for the sake of contradiction, that $\succeq$ is not acyclic. Cyclicity of $\succeq$ implies the existence of a finite subset $\bar{X} \subseteq X$ that does not admit a maximal element. By Lemma 1, the corresponding symmetric two-player zero-sum game $(\bar{X}, \operatorname{sgn}(\bar{v}))$ therefore has no pure-strategy equilibrium.

According to Theorem 7 in Duersch et al. (2012a), this absence of a pure equilibrium implies that the game $(\bar{X}, \operatorname{sgn}(\bar{v}))$ is not quasiconcave. Consequently, it is impossible to define a total order on $\bar{X}$ that renders the game quasiconcave. Extending this argument, it follows that no total order on the larger set X can make (X,sgn(v))quasiconcave.

Therefore, if (X, v) is weakly quasiconcave, its equivalent decision problem must be acyclic. This completes the proof [28].

Theorem 2 Let (X, v) be a symmetric two-player zero-sum game. If (X, v) is **weakly quasiconcave**, then the game admits at least one **pure-strategy equilibrium**.

Proof. By Lemma 4, if the game G = (X, v) is weakly quasiconcave, then the associated preference relation ≥ in its equivalent decision problem $D_G = (X, \succeq)$ is acyclic. Applying Lemma 2, acyclicity ensures the existence of a maximal element, which directly corresponds to a pure-strategy equilibrium in the original game G.

RELATIONS BETWEEN SUFFICIENT CONDITIONS

In this section, we first examine the relationships among the well-established sufficient conditions for the existence of pure equilibria in symmetric two-player zero-sum games. Next, we explore the interconnections between the newly introduced conditions, such as interchangeability and weak quasiconcavity. Finally, we present results that formally establish links between the classical and new conditions, highlighting how the novel conditions extend or refine existing frameworks.

Concluding Remarks: This study makes several contributions to the theory of symmetric zero-sum games by investigating sufficient conditions for the existence of pure-strategy equilibria. First, we introduced two new sufficient conditions: interchangeability and a weakened form of quasiconcavity, extending the concept originally proposed by Duersch et al. (2012a). Second, we analyzed the relationships between classical and newly introduced conditions, employing a framework grounded in the preference relations naturally induced by symmetric zero-sum games.

For future research, it would be worthwhile to explore whether the framework and conditions developed here can be generalized to broader classes of games, including asymmetric zero-sum games and symmetric non-zero-sum games, potentially providing further insights into the existence and structure of equilibria in more complex strategic settings.

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